

Nuclear Reactions: Lecture on two-body description

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Contents

- Fundamentals: Types of nuclear reactions
- Observables
- Compound vs Direct reactions
- Collision theory-elastic scattering
 - Lab vs CM coordinate systems
- Effective potential
- Born and DWBA
 - examples
- Optical model
- Coupled-channels model-Inelastic scattering

Suggested books

G.R. Satchler,
Introduction to Nuclear Reactions, Oxford Uni. Press.

I. Thompson & F. Nunes,
Nuclear Reaction for Astrophysics, Cambridge Uni. Press.

C. Bertulani & P. Danielewicz,
Introduction to Nuclear Reactions, Taylor&Francis

L.S. Rodberg & R.M. Thaler,
Introduction to Quantum Scattering Theory, Academic

Advanced Level;

G.R. Satchler, Direct Nuclear Reactions, Oxford Uni. Press

N. Austern, Direct Nuclear Reactions Theories, Wiley

N. K. Glendenning, Direct Nuclear Reactions, World Scientific

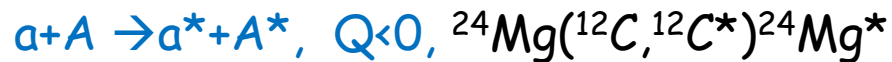
Types of Nuclear Reactions

Scattering vs Reaction

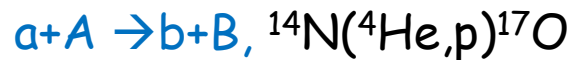
1- Elastic Scattering (always present) , No energy change $X(a,a)X$



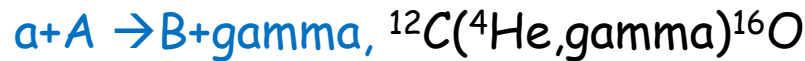
2- Inelastic Scattering , Energy change $X(a,a^*)X^*$



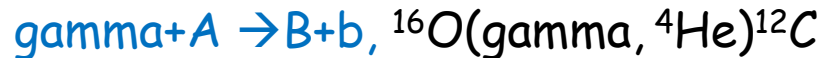
3- Inelastic Reaction, Energy and nucleon change



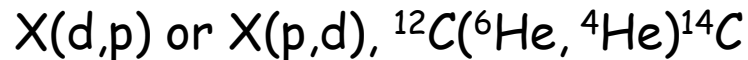
4- Radiative Capture



5- Photo Reaction



6- Transfer Reaction



$$Q = (m_{\text{initial}} - m_{\text{final}})c^2$$

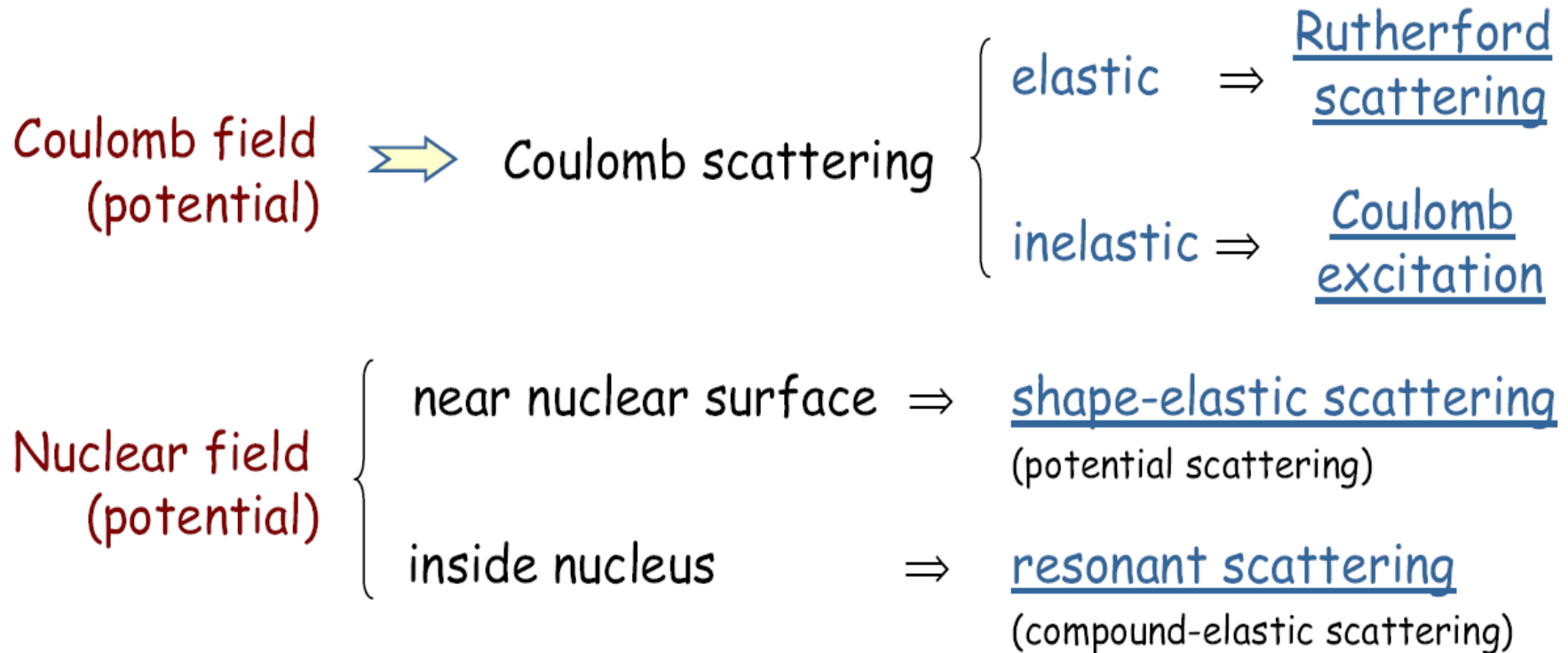
Observables

Experiment:

- **Angular distribution**: E fixed, θ variable
- **Excitation function**: θ fixed, E variable

Theory:

- “**Direct**” problem: determine cross-section from the potential
- “**Inverse**” problem : determine the potential V from cross-section



Nuclear Reactions:

Compound

- Small b values (head-on)
- Low energy
- Two step: No relation to entrance channel



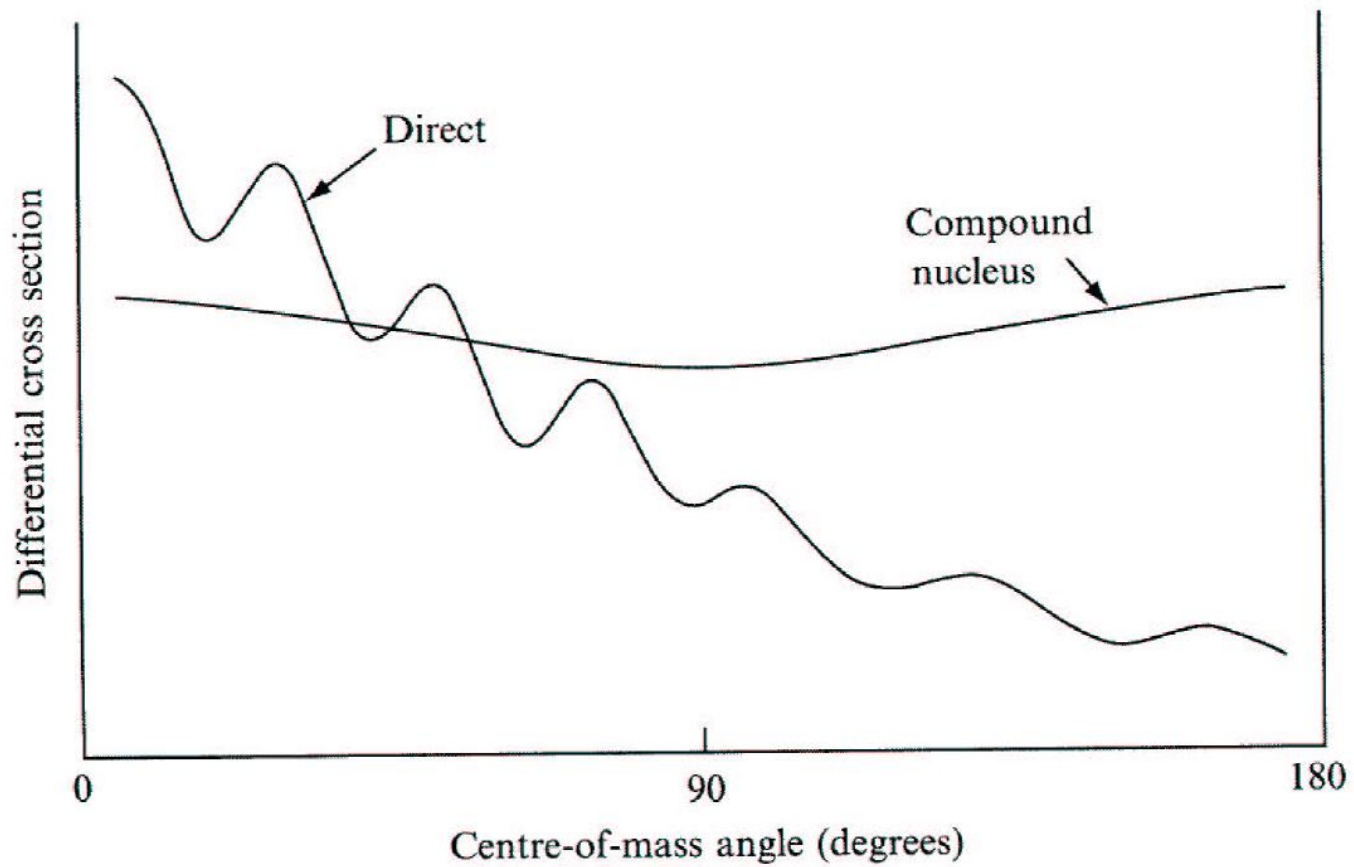
- Timescale: 10^{-16} - 10^{-18} s
- Isotropic/Symmetric X-section ($\sim\theta=90^\circ$)

Direct

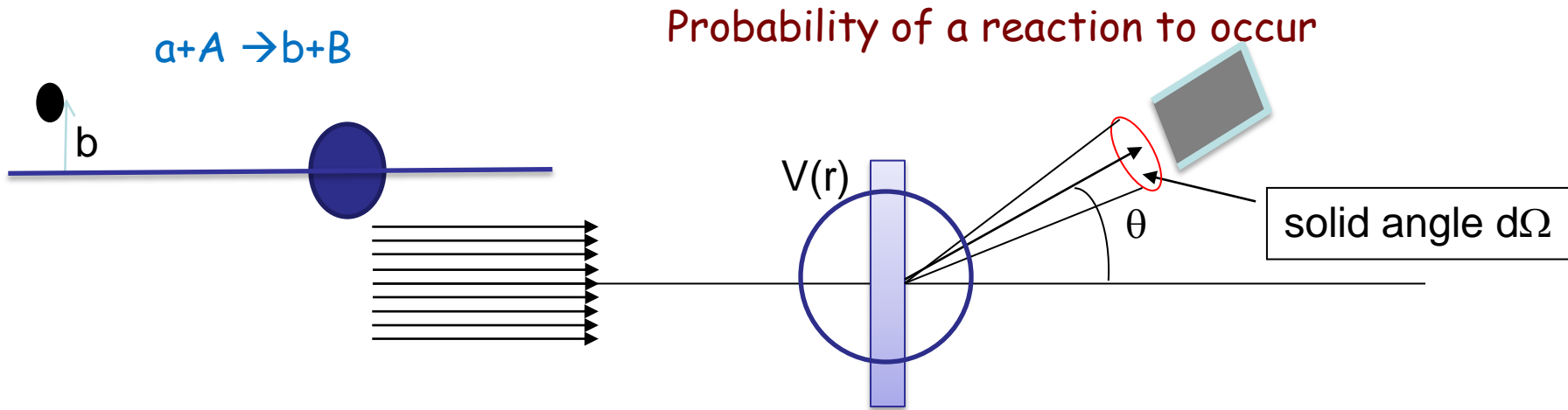
- Large b values (peripheral)
- High energy
- One step



- Timescale: 10^{-22} s
- Forward peaked-Oscillatory X-section



Cross-section:



I_a = current of incident particles $a \Rightarrow$ no. particles / unit time

N = no. target nuclei / unit area

R_b = no. detected particles b / unit time ("rate")

Proportionality relation: $R_b \sim I_a N$

Define:

cross section σ

$$\sigma = \frac{R_b}{I_a N}$$

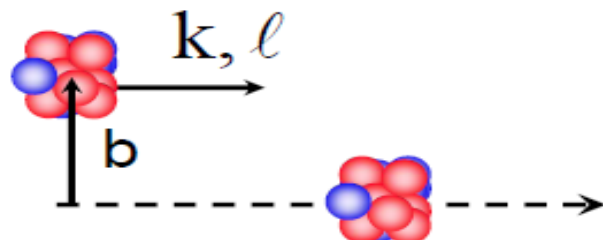
cross section has dimension of AREA

unit for cross section σ :

$$\text{barn} = 10^{-24} \text{ cm}^2 = 100 \text{ fm}^2$$

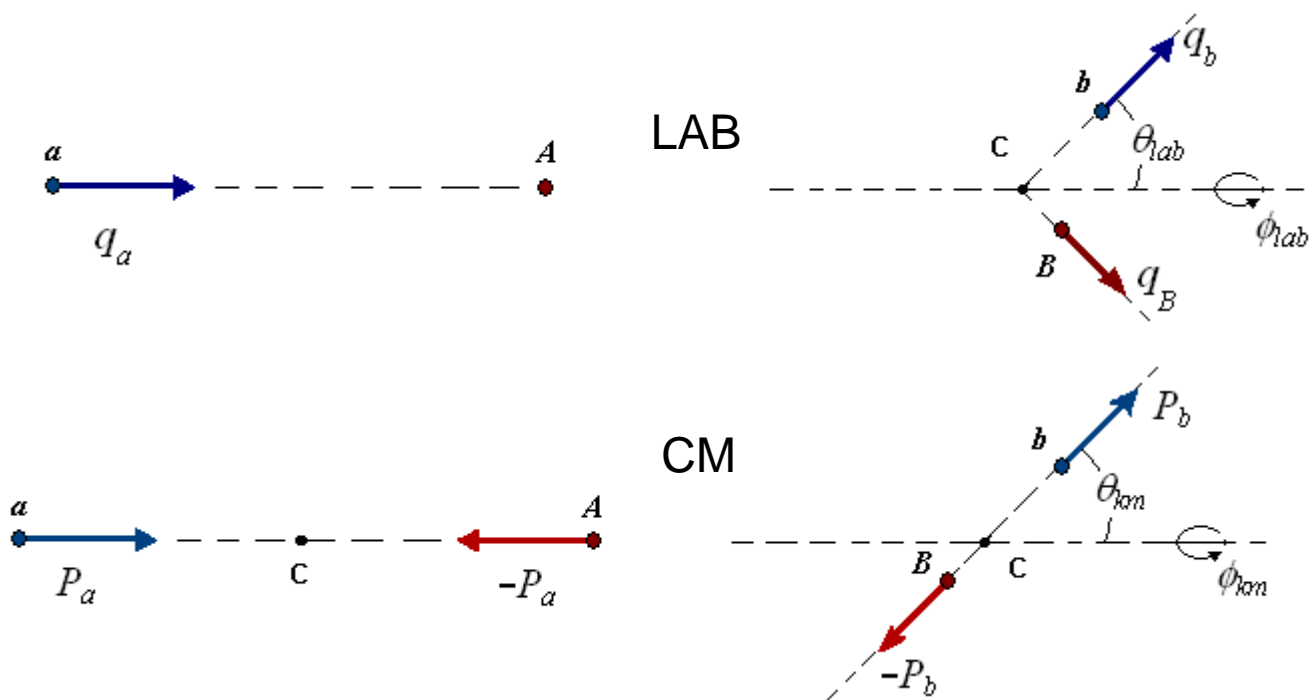
Collision theory: elastic scattering

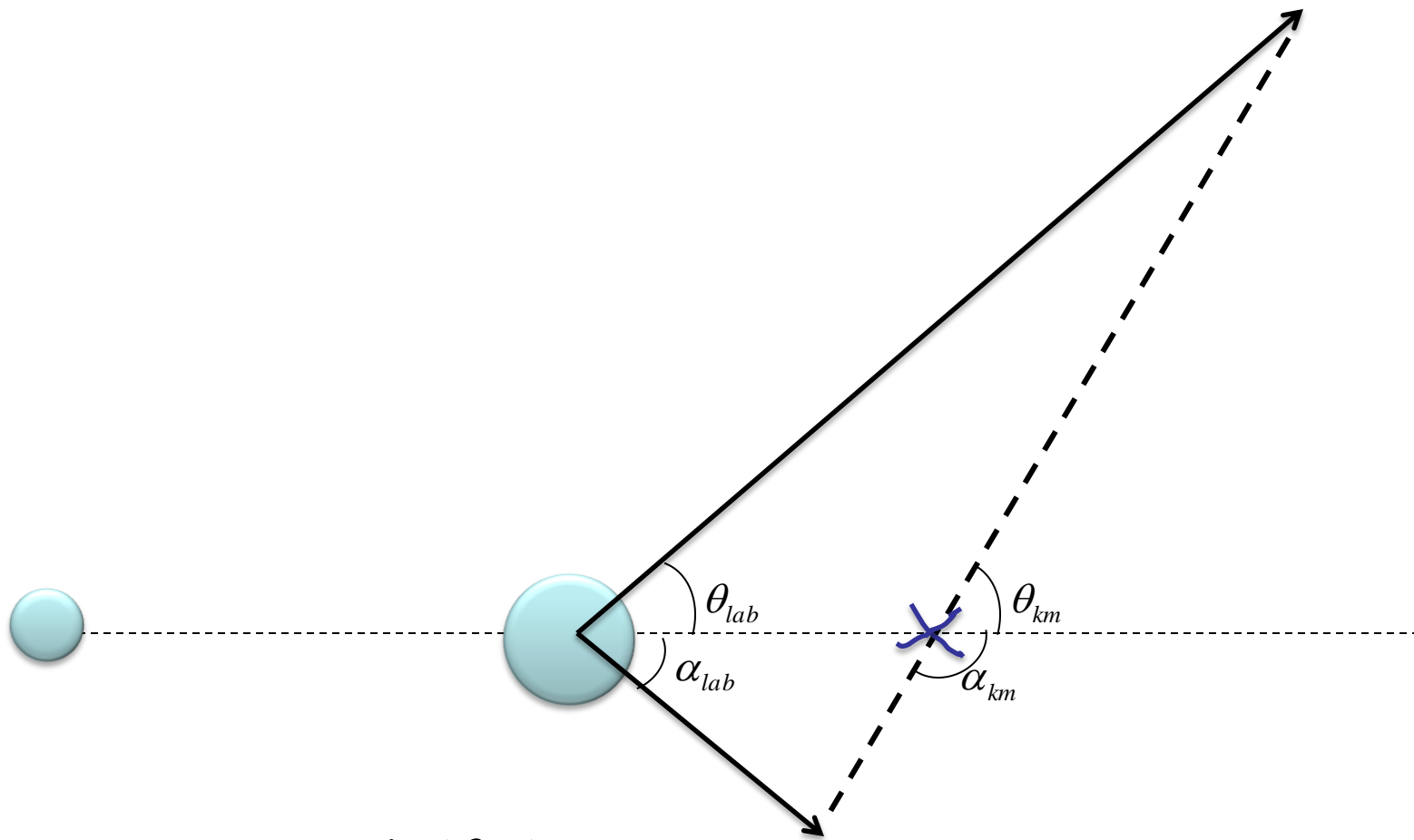
b =impact parameter



Before collision

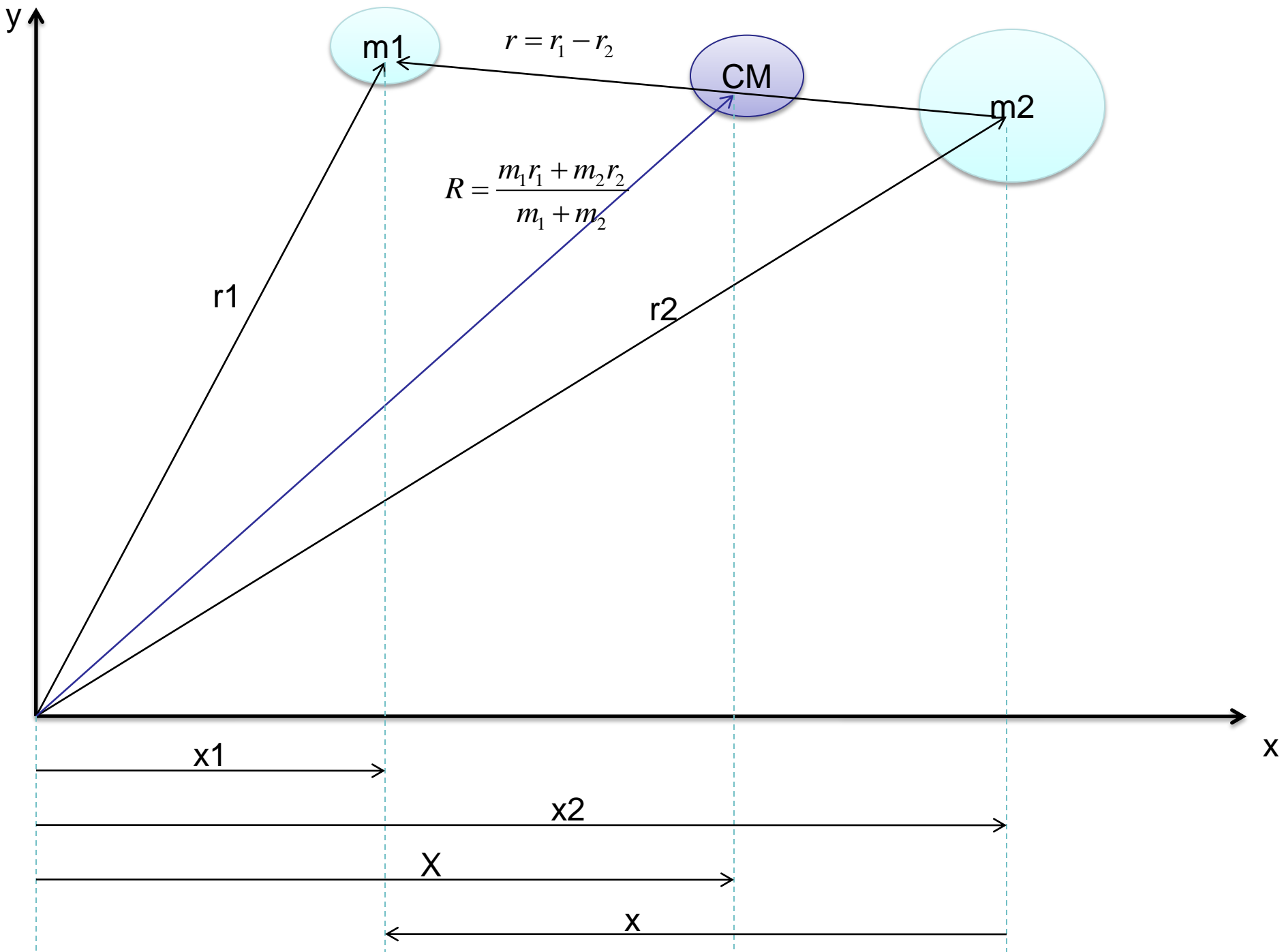
After collision





$$\tan(\theta_{lab}) = \frac{\sin(\theta_{cm})}{\cos(\theta_{cm}) + v_M / v_{cm}}$$

$$E_{CM} = \frac{m_2}{m_1 + m_2} E_{lab}$$



$$\left(-\frac{\hbar^2}{2m_1} \nabla_1^2 - \frac{\hbar^2}{2m_2} \nabla_2^2 + V(r_1, r_2) \right) \psi = E\psi$$

$$\nabla_i \psi = \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} + \frac{\partial \psi}{\partial z_i}$$

$$\left(-\frac{\hbar^2}{2[m_1 + m_2]} \nabla_R^2 - \frac{\hbar^2}{2[m_1 m_2 / (m_1 + m_2)]} \nabla_r^2 + V(r) \right) \psi = E\psi$$

$$-\frac{\hbar^2}{2M} \nabla_R^2 \psi - \left(\frac{\hbar^2}{2m} \nabla_r^2 + V(r) \right) \psi = E\psi$$

R

Center of Mass

r

Relative Motion

$$-\frac{\hbar^2}{2M} \nabla_R^2 \psi = \epsilon_0 \psi$$

$$-\left(\frac{\hbar^2}{2m} \nabla_r^2 + V(r) \right) \psi = \epsilon \psi$$

$$\frac{\partial \psi}{\partial x_1} = \frac{\partial \psi}{\partial X} \frac{\partial X}{\partial x_1} + \frac{\partial \psi}{\partial x} \frac{\partial x}{\partial x_1}$$

$$\frac{\partial X}{\partial x_1} = \frac{m_1}{m_1 + m_2}$$

$$\frac{\partial x}{\partial x_1} = 1$$

$$\frac{\partial \psi}{\partial x_1} = \frac{m_1}{m_1 + m_2} \frac{\partial \psi}{\partial X} + \frac{\partial \psi}{\partial x}$$

$$m = m_1 m_2 / (m_1 + m_2)$$

$$M = m_1 + m_2$$

Two body problem: Schrödinger Equation with effective potential

Two Body Problem

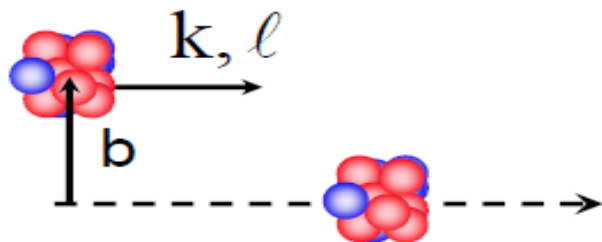
1. Move from LAB system to CM.
2. Separate the CM motion
3. Find the reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

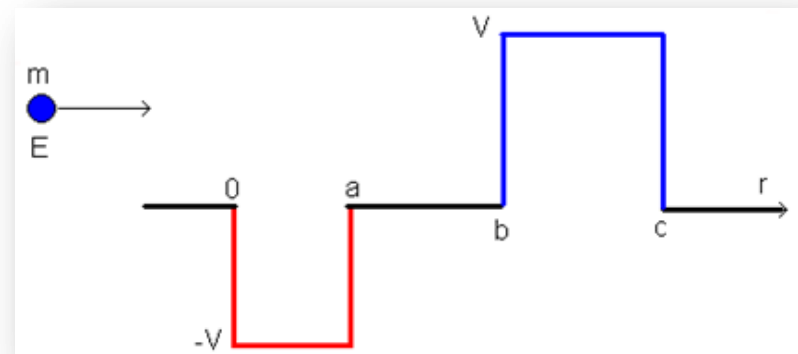
4. Define the total Central Potential $V(r)$
5. Solve the Sch. Eq. For this $V(r)$.

“Reducing the interaction of projectile and target nuclei to interaction of the reduced mass with a potential between them.”

b = impact parameter



=

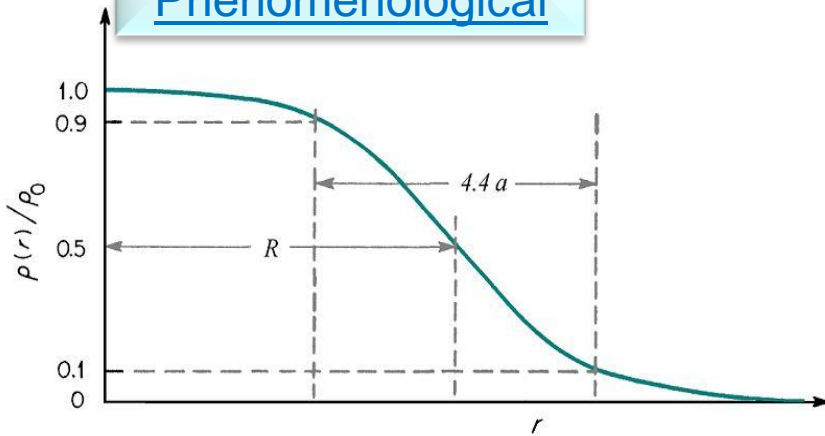


Effective Potential

$$V(r) = V_N(r) + \frac{Z_1 Z_2 e^2}{r} + \frac{\hbar^2 l(l+1)}{2\mu r^2}$$

Nuclear Potentials

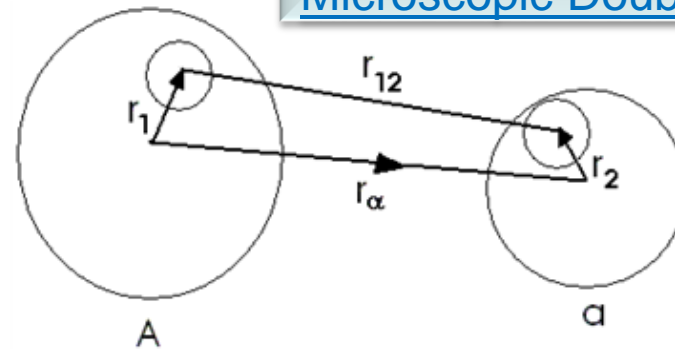
Phenomenological



$$V(r) = \frac{-V_0}{1 + \exp\left[\frac{r-R}{a}\right]^n}$$

$$R = r_0 A^{1/3}$$

Microscopic Double Folding



$$U_F(\vec{R}) = \int d\vec{r}_1 \int d\vec{r}_2 \rho_1(\vec{r}_1) \rho_2(\vec{r}_2) v(\vec{R} - \vec{r}_1 + \vec{r}_2)$$

$\rho_1(\vec{r}_1) \rightarrow$ Density of A

$\rho_2(\vec{r}_2) \rightarrow$ Density of a

$v(\vec{R} - \vec{r}_1 + \vec{r}_2) \rightarrow$ NN interaction + exchange term

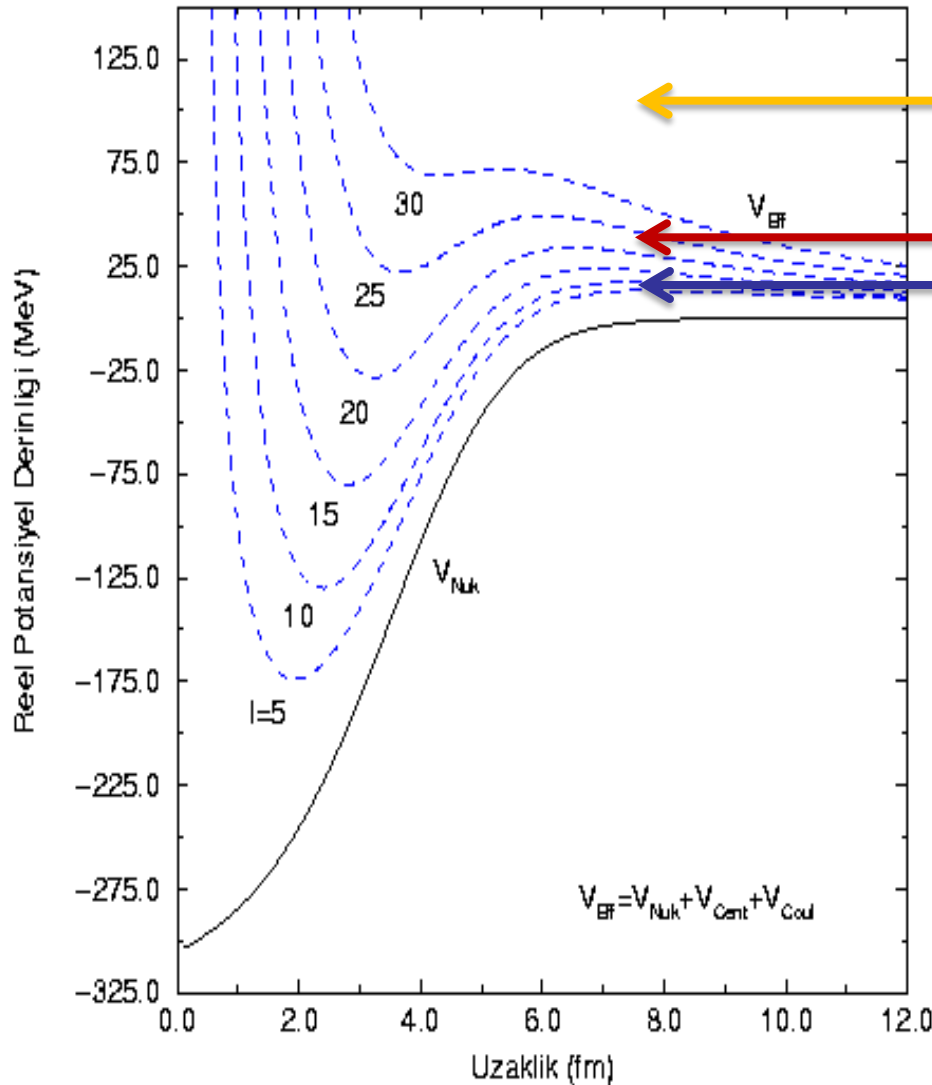
$$v(r) = 7999 \frac{\exp(-4r)}{4r} - 2134 \frac{\exp(-2.5r)}{2.5r} + J_{00}(E) \delta(r)$$

$$V(r) = V_N(r) + \frac{Z_1 Z_2 e^2}{r} + \frac{\hbar^2 \ell(\ell + 1)}{2\mu r^2}$$

Coulomb Barrier vs Energy

$$R_B \sim A_1^{1/3} + A_2^{1/3}$$

$$V_B = 1.44 Z_1 Z_2 / R_B$$



Below the barrier

Very few ℓ values

R matrix

Nuclear
Astrophysics

Near the barrier

Limited ℓ values \rightarrow partial wave expansion
ex: Optical, CC, CDCC, DWBA, etc.

Far above the barrier

(Too) many ℓ values
 \rightarrow no partial wave expansion
ex: Eikonal, Glauber, semi-classic theories, etc.

Two body problem: Numerical Solution of Schrödinger Equation

Analytic solution of the Schrödinger equation is limited to few potentials: SW, HO

In general, there is no **analytic solution** → **numerical approach**

$$-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} u_\ell(r) + (V(r) - E)u_\ell(r) = 0$$

$$u_\ell(r) \rightarrow F_\ell(kr, \eta) \cos \delta_\ell + G_\ell(kr, \eta) \sin \delta_\ell$$

Numerical solution : discretization N points, with mesh size h

- $u_\ell(0)=0$, $u_\ell(h)=1$ (or any constant)
- $u_\ell(2h)$ is determined numerically from $u_\ell(0)$ and $u_\ell(h)$ (Numerov algorithm)
- $u_\ell(3h), \dots u_\ell(Nh)$
- for large r: matching to the asymptotic behaviour → phase shift

Bound states ($E < 0$): same idea

Born and Distorted Wave Born Approximation

$$L_k(r)\psi(r) = U(r)\psi(r) \quad \text{where} \quad L_k(r) = \nabla^2 + k^2$$

Multiplying by $L_k^{-1}(r)$ and integrating all over the space we get

$$\psi(r) = \phi_k(r) + \int U(r')\psi(r')L_k^{-1}(r)\delta(r' - r)dr'$$

$\phi_k(r) = e^{ikr}$ is the free particle solution ($V=0$). Using Green functions:

$$L_k^{-1}(r)\delta(r' - r) = G_k(r - r') \quad G_k^+(r - r') = -\frac{1}{4\pi} \frac{e^{ik|r-r'|}}{|r - r'|}$$

$$\psi_k^+(r) = \phi_k(r) - \frac{1}{4\pi} \int \frac{e^{ik|r-r'|}}{|r - r'|} U(r')\psi_k^+(r')dr' \quad \frac{1}{r - r'} \approx \frac{1}{r} \quad k|r - r'| \approx kr - k \cdot r'$$

$$\psi_k^+(r) = \phi_k(r) - \frac{e^{ikr}}{4\pi r} \int e^{-ik' \cdot r'} U(r')\psi_k^+(r')dr'$$

Scattering Amplitude and Cross-section

$$f(\theta, \varphi) = -\frac{1}{4\pi} \int e^{-ik'.r'} U(r') \psi_k^+(r') dr'$$

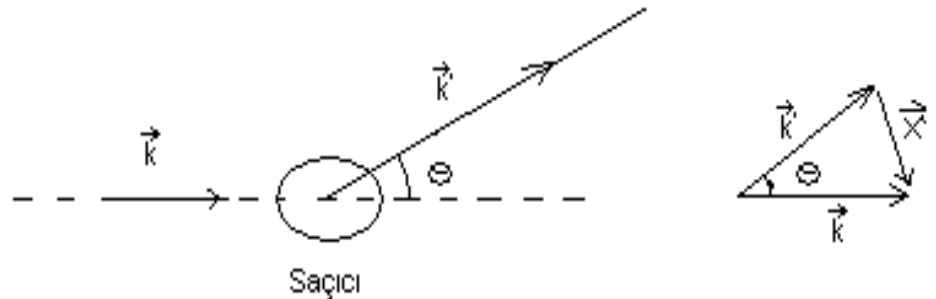
If we use plane wave for the $\psi_k^+(r)$ scattering amplitude in Born Approximation is:

$$f_{BA}(\theta, \varphi) = -\frac{1}{4\pi} \int e^{-ik'.r'} U(r') e^{ik.r'} dr'$$

$$q = k - k'; \quad f_{BA}(\theta, \varphi) = -\frac{1}{4\pi} \int U(r) e^{iqr} d^3r$$

For spherically symmetric potential $\int U(r) e^{iqr} d^3r = \frac{4\pi}{q} \int U(r) r \sin(qr) dr$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \quad \sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$



Distorted Wave Born Approximation

$U = U_1 + U_2$ such that $U_1 > U_2$

$$[\nabla^2 + k^2 - U_1(r)]\chi_1(k, r) = 0$$

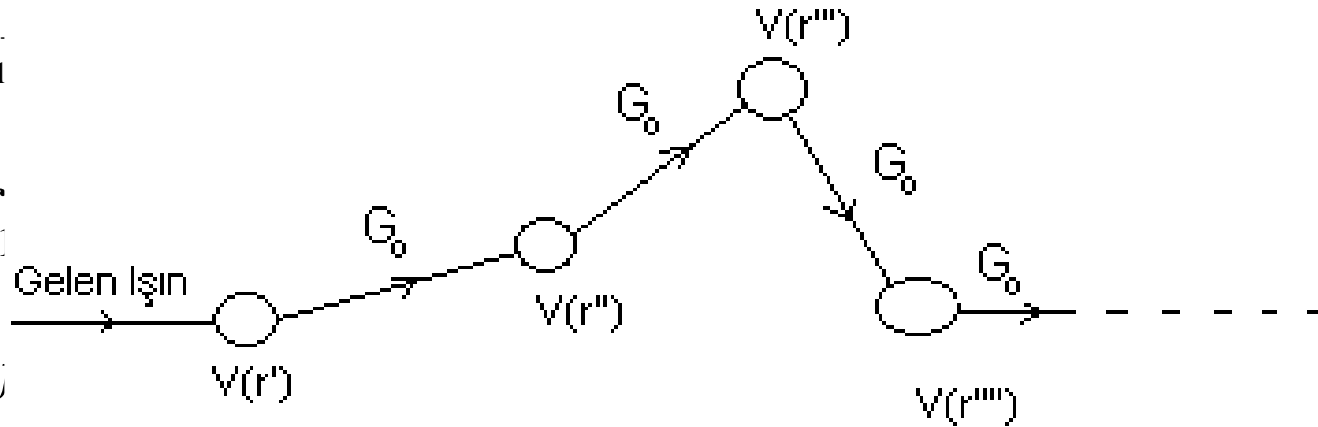
$$\chi_1(k, r) = \chi_1^-(k, r) + \chi_1^+(k, r)$$

Incoming and outgoing waves

$$\chi(k, r) \xrightarrow{r \rightarrow \infty} \chi_1$$

$$f(\theta, \varphi) = f_1$$

$$f_{DWBA}(\theta, \varphi) = f_2$$



$$(E - H_0)\psi = V\psi \quad \implies \psi = (E - H_0)^{-1}V\psi = G_0(E)V\psi$$

$$\psi = \phi + G_0V\phi + G_0VG_0V\phi + \dots$$

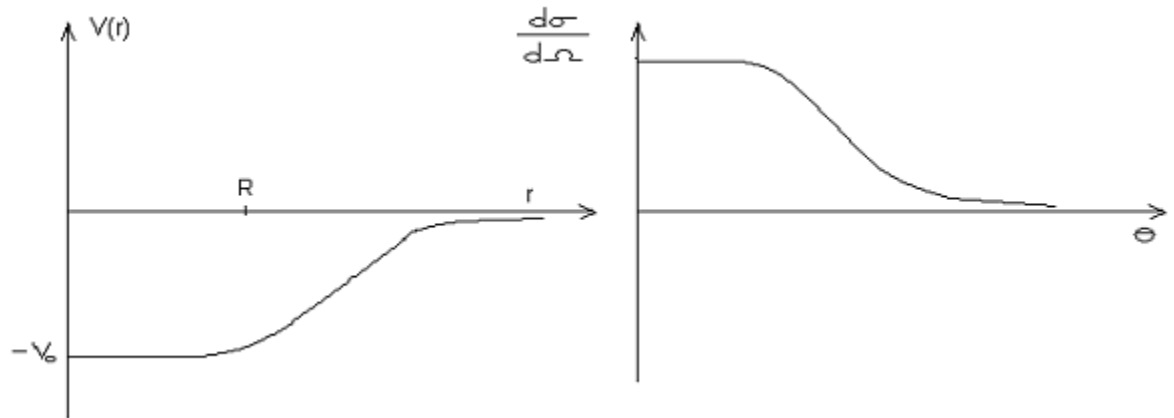
$$f(\theta, \varphi) = -\frac{m}{2\pi\hbar^2} \left[\int dr e^{-i\vec{k}' \cdot \vec{r}} V(r) e^{i\vec{k} \cdot \vec{r}} + \int dr \int dr' e^{-i\vec{k}' \cdot \vec{r}} V(r) G_0(r, r') V(r') e^{i\vec{k} \cdot \vec{r}'} + \int dr \int dr' \int dr'' + \dots \right]$$

Example: Gaussian Potential

$$V(r) = -V_0 e^{-\left(\frac{r}{R}\right)^2} \quad f(\theta) = \int_0^{\infty} V(r) \frac{\sin qr}{qr} 4\pi r^2 dr \quad q = 2k \sin\left(\frac{\theta}{2}\right)$$

$$f(\theta) = -V_0 \int_0^{\infty} e^{-\left(\frac{r}{R}\right)^2} \frac{\sin qr}{qr} 4\pi r^2 dr = -(2\pi)^{\frac{3}{2}} V_0 R^3 e^{-\frac{(qR)^2}{2}}$$

$$\frac{d\sigma}{d\Omega} = C e^{-(2kR)^2 \sin^2\left(\frac{\theta}{2}\right)}$$



Schrödinger equation II: Partial wave methods

We must solve:

$$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + \frac{2\mu}{\hbar^2} [E_{cm} - U_{\ell j}(r)] \right) u_{\ell j}(r) = 0$$

For bound states $E_{cm} < 0$ $\kappa_b = \sqrt{\frac{2\mu|E_{cm}|}{\hbar^2}}$

$$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2} U_{\ell j}(r) - \kappa_b^2 \right) u_{n\ell j}(r) = 0$$



Discrete
Spectrum

For scattering states $E_{cm} > 0$ $k = \sqrt{\frac{2\mu E_{cm}}{\hbar^2}}$

$$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2} U_{\ell j}(r) + k^2 \right) u_{k\ell j}(r) = 0$$



Continuous
Spectrum

General Solution:

$$U_l(r) = F_l(r) + iG_l(r) + S_l[G_l(r) - iG_l(r)]$$

$$F_l(r) = krj_l(kr) \quad \text{Bessel functions}$$

$$G_l(r) = -kr\eta_l(kr) \quad \text{Neumann functions}$$

$$f(\theta) = f_c(\theta) + \frac{1}{2ik} \sum_{l=0}^{l=\infty} (2l+1)(S_l - 1)e^{2i\sigma_l} P_l(\cos \theta) \quad \text{scattering amplitude}$$

$$\sigma_{el} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) |1 - S_\ell|^2$$



$$\sigma_{tot} = \sigma_{el} + \sigma_R$$

$$\sigma_R = \frac{\pi}{k^2} \sum_l (2l+1) [1 - |S_l|^2]$$

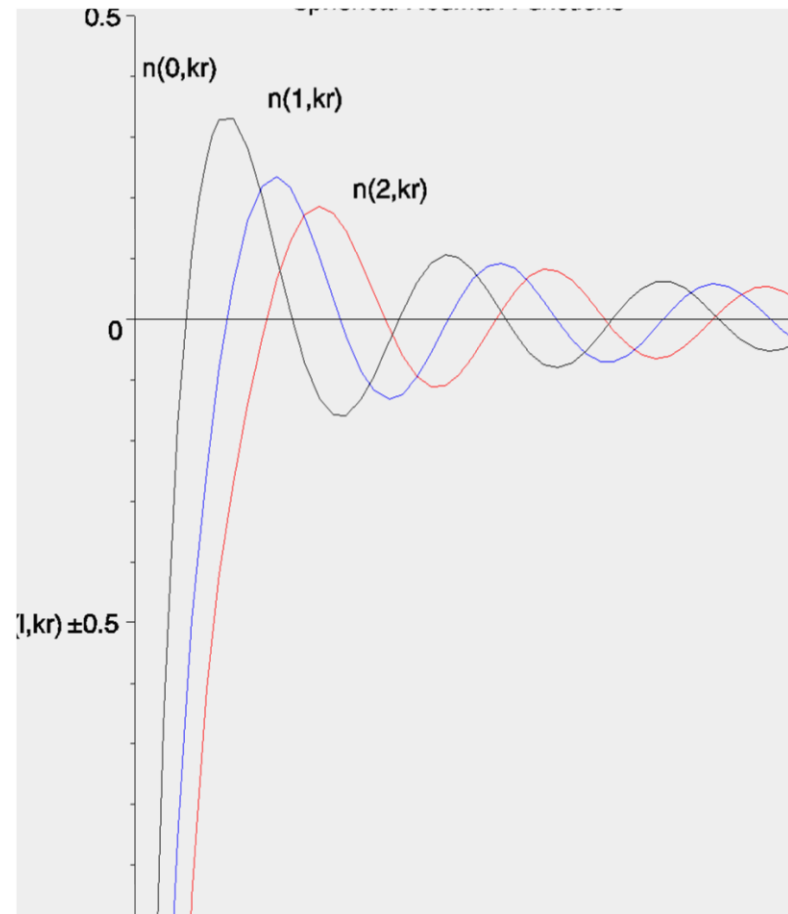
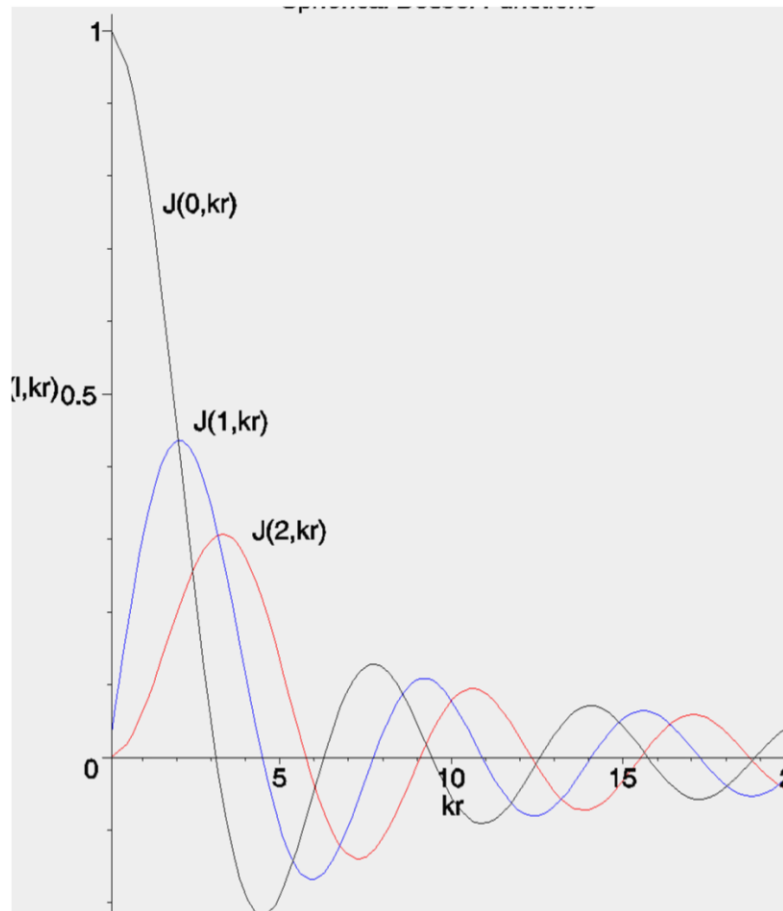
Bessel and Neumann functions

For small x $j_l(x) \rightarrow \frac{x^l}{(2l+1)!!}$

$n_l(x) \rightarrow -\frac{(2l-1)!!}{x^{l+1}}$

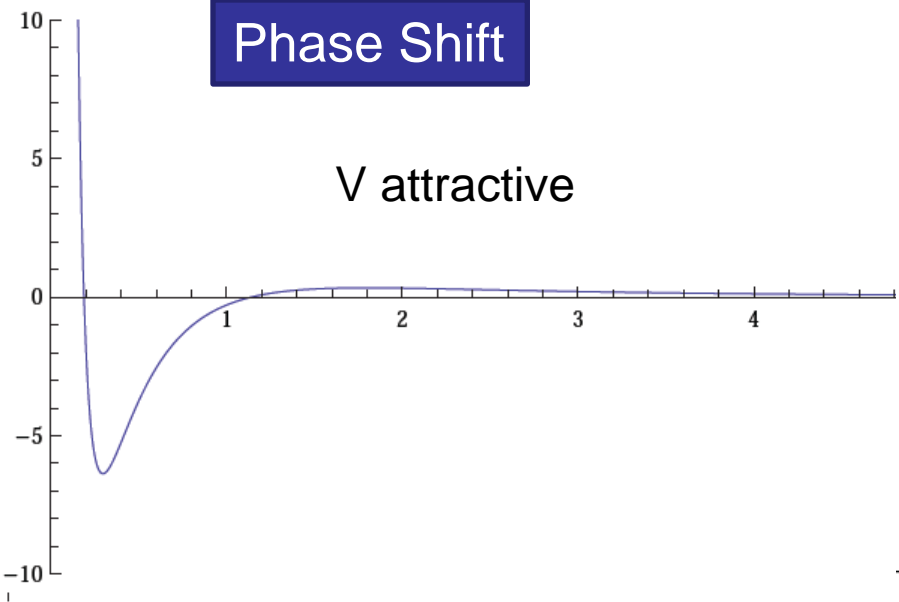
For large x $j_l(x) \rightarrow \frac{1}{x} \sin(x - l\pi/2)$

$n_l(x) \rightarrow -\frac{1}{x} \cos(x - l\pi/2)$

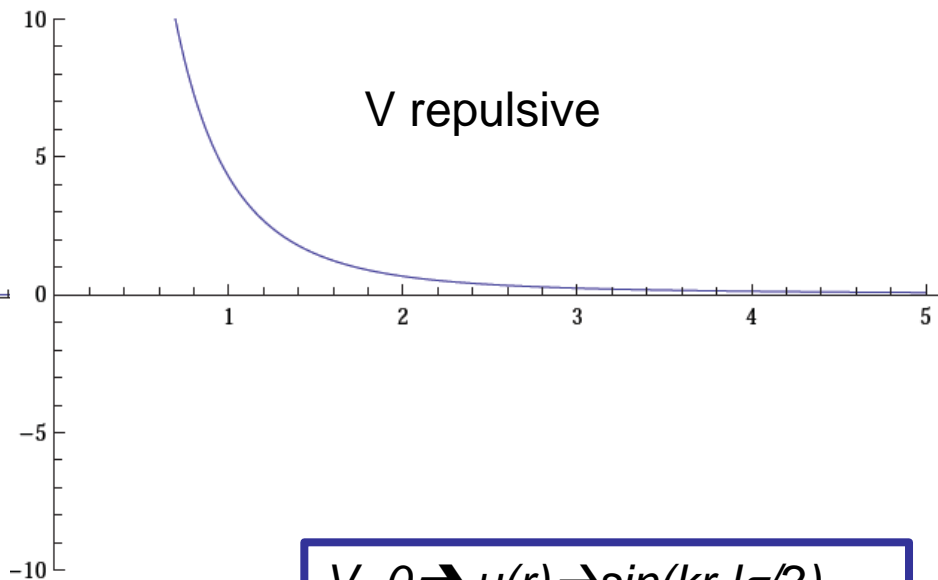


Phase Shift

V attractive

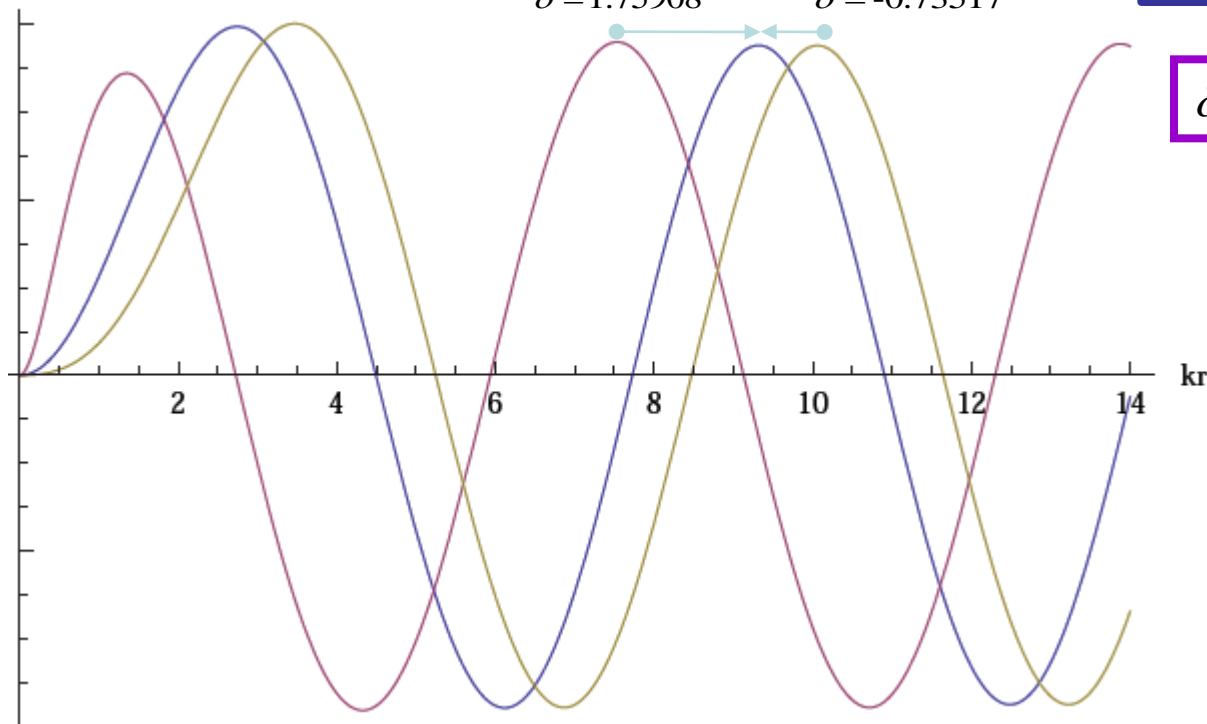


V repulsive



$$V=0 \rightarrow u(r) \rightarrow \sin(kr - l\pi/2)$$
$$\delta_l = 0$$

$$\delta = 1.75908$$
$$\delta = -0.73517$$

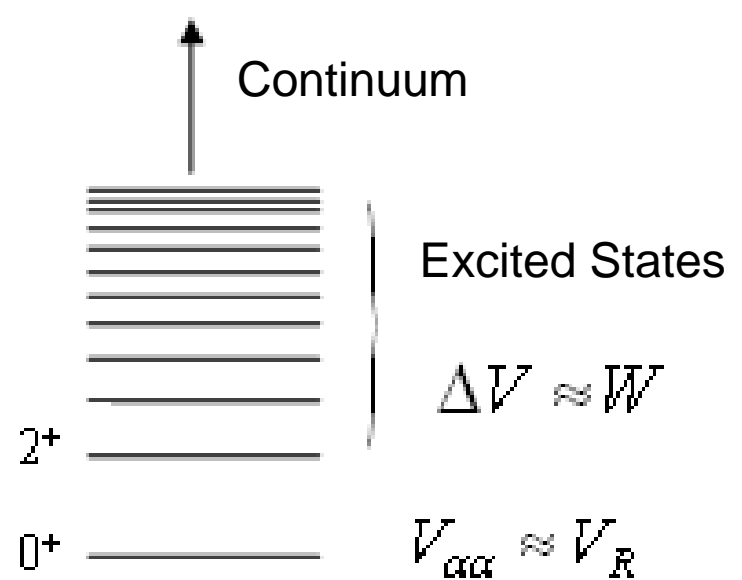


$$\delta_l > 0 \rightarrow V \text{ attractive}$$

$$V \neq 0 \rightarrow u(r) \rightarrow \sin(kr - l\pi/2 + \delta_l)$$

$$\delta_l < 0 \rightarrow V \text{ repulsive}$$

Optical Model



- Feshbach's formalism

$$V_{Nuclear} = V_{\alpha\alpha} + V_{\alpha\beta} \frac{1}{E^{(+)} - H_{\beta\beta}} V_{\beta\alpha} = V_{\alpha\alpha} + \Delta V(E) \approx V_R + iW$$

- Nuclear Potential:
 - Complex
 - Non-local
 - Energy and model space dependent
 - Resonant...

Model: General

- Optical Model (Elastic Scattering)

$$\left[\frac{\hbar^2}{2\mu r} \left(\frac{d^2}{dr^2} + k^2 - \frac{L(L+1)}{r^2} \right) - (\alpha SLJ | V_\alpha | \alpha SLJ) \frac{1}{r} - \frac{V_C}{r} \right] \chi_{p,p}^J(k, r) = 0$$

- Coupled-Channels Model (Elastic+Inelastic)

$$\left[\frac{\hbar^2}{2\mu r} \left(\frac{d^2}{dr^2} + k'^2 - \frac{L'(L'+1)}{r^2} \right) - (\alpha' S' L' J | V_\alpha | \alpha' S' L' J) \frac{1}{r} - \frac{V_C}{r} \right]$$

$$\chi_{p',p}^J(k', r) = \sum_{p'' \neq p'} (\alpha' S' L' J | V_\alpha | \alpha'' S'' L'' J) \frac{1}{r} \chi_{p'',p}^J(k'', r)$$

where $p \equiv \alpha LS$, $p' \equiv \alpha' L' S'$ $(\alpha SLJ | V_\alpha | \alpha SLJ) = V_{\alpha\alpha}$, $(\alpha' S' L' J | V_\alpha | \alpha'' S'' L'' J) = V_{\alpha\beta}$

- Deformation (Rotationel)

$$R = R_0 [1 + \beta_1 Y_{20}(\theta_1, \phi_1) + \beta_2 Y_{20}(\theta_2, \phi_2)]$$



$\beta < 0$



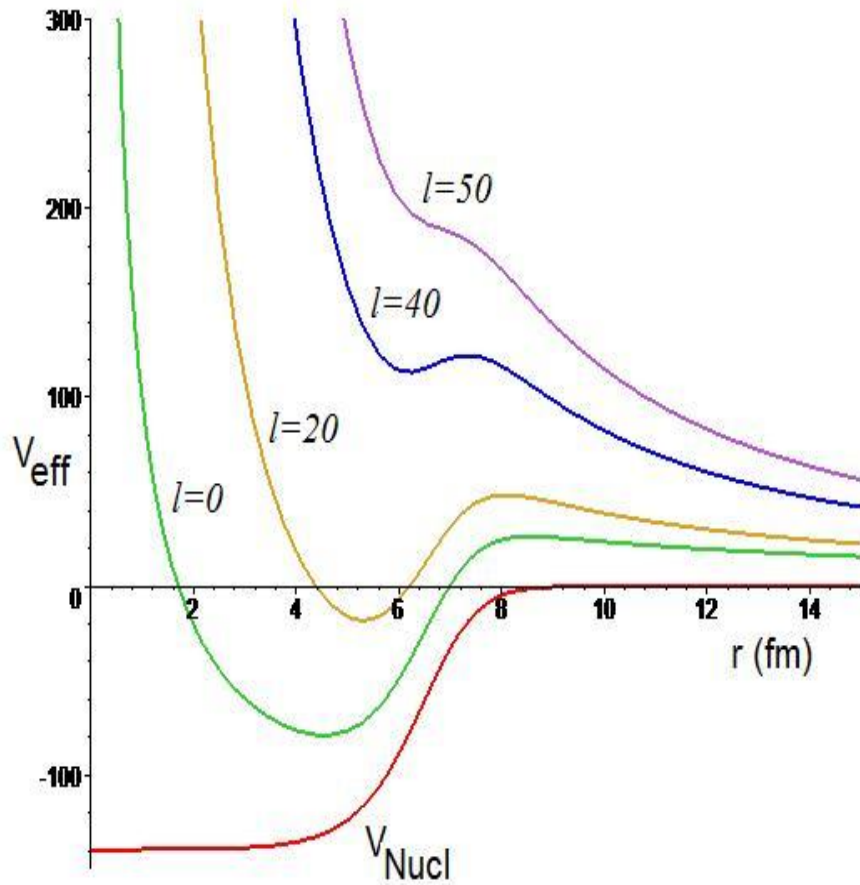
$\beta = 0$



$\beta > 0$

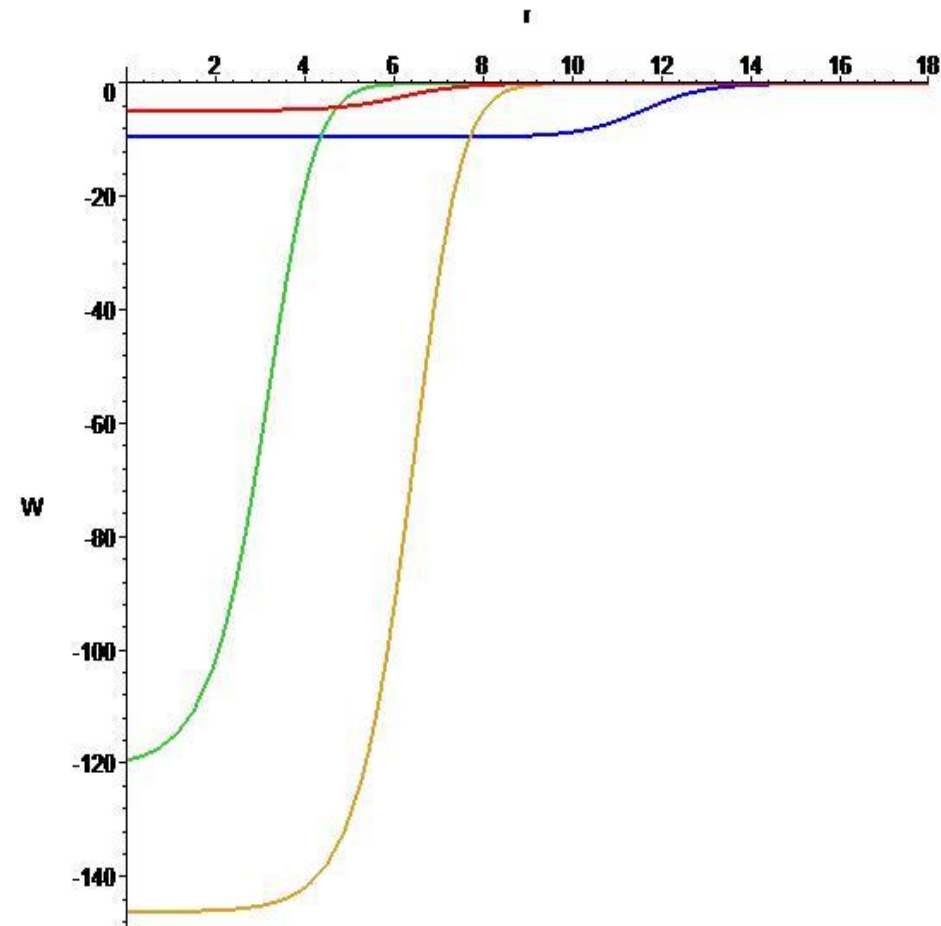
Optical Potentials

Real & Effective Real Potentials



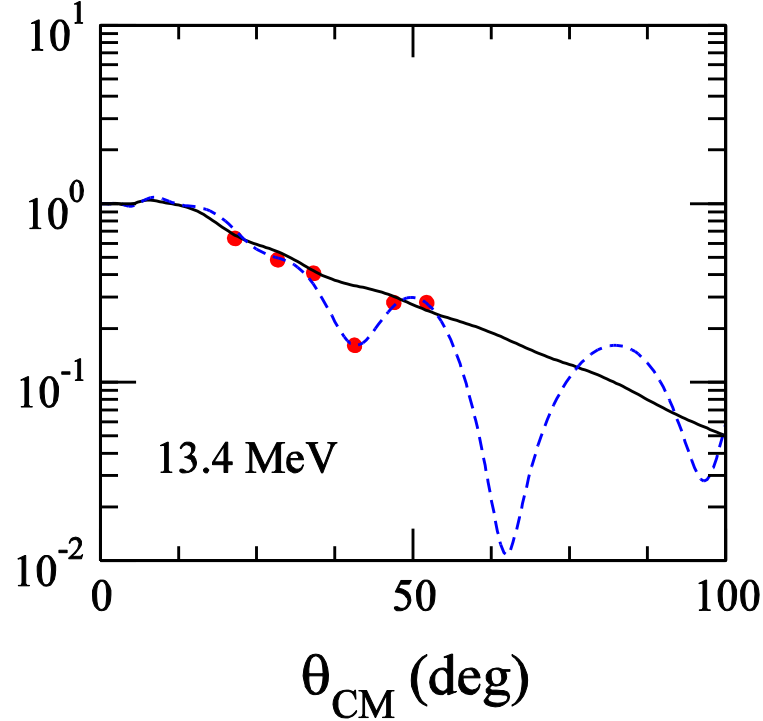
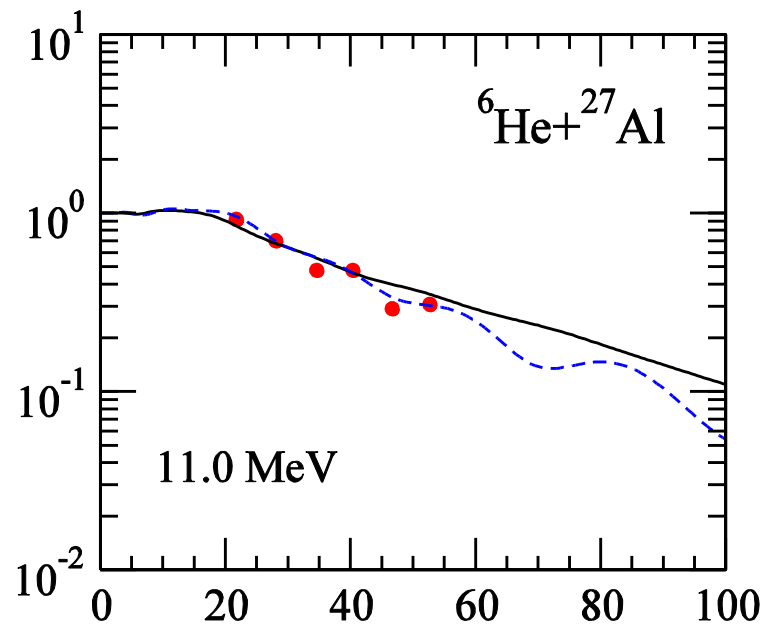
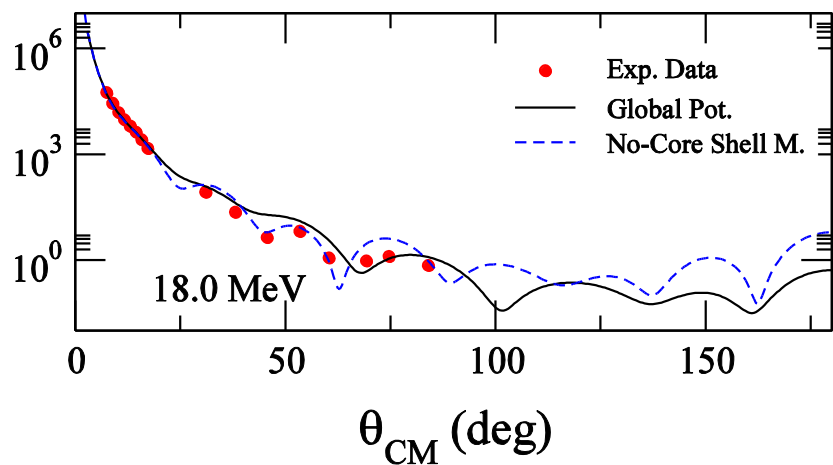
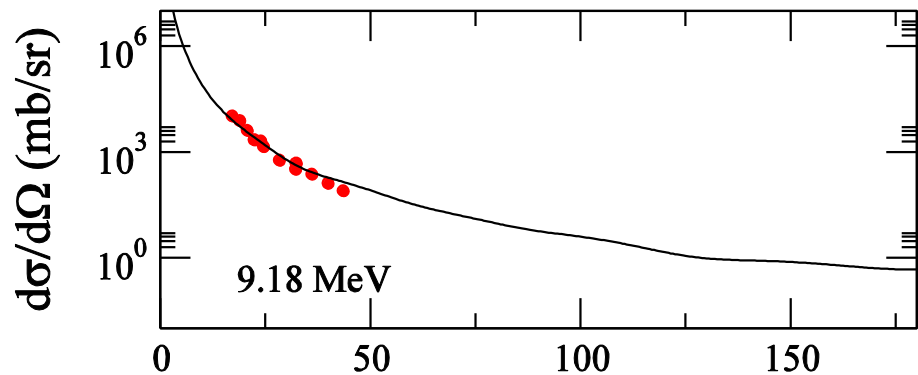
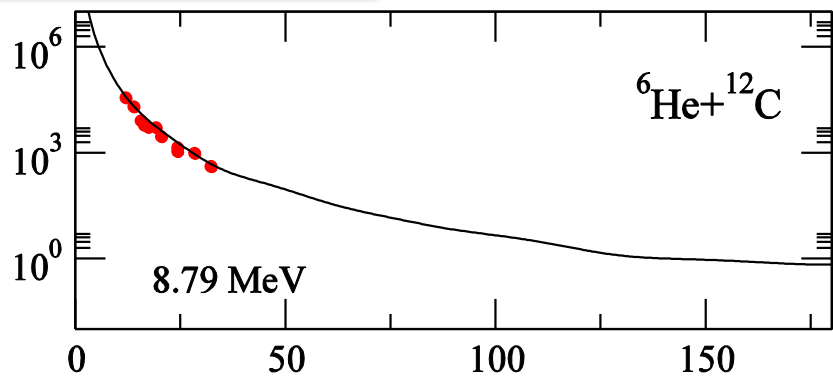
${}^6\text{He} + {}^{208}\text{Pb}$ @ 18.0 MeV

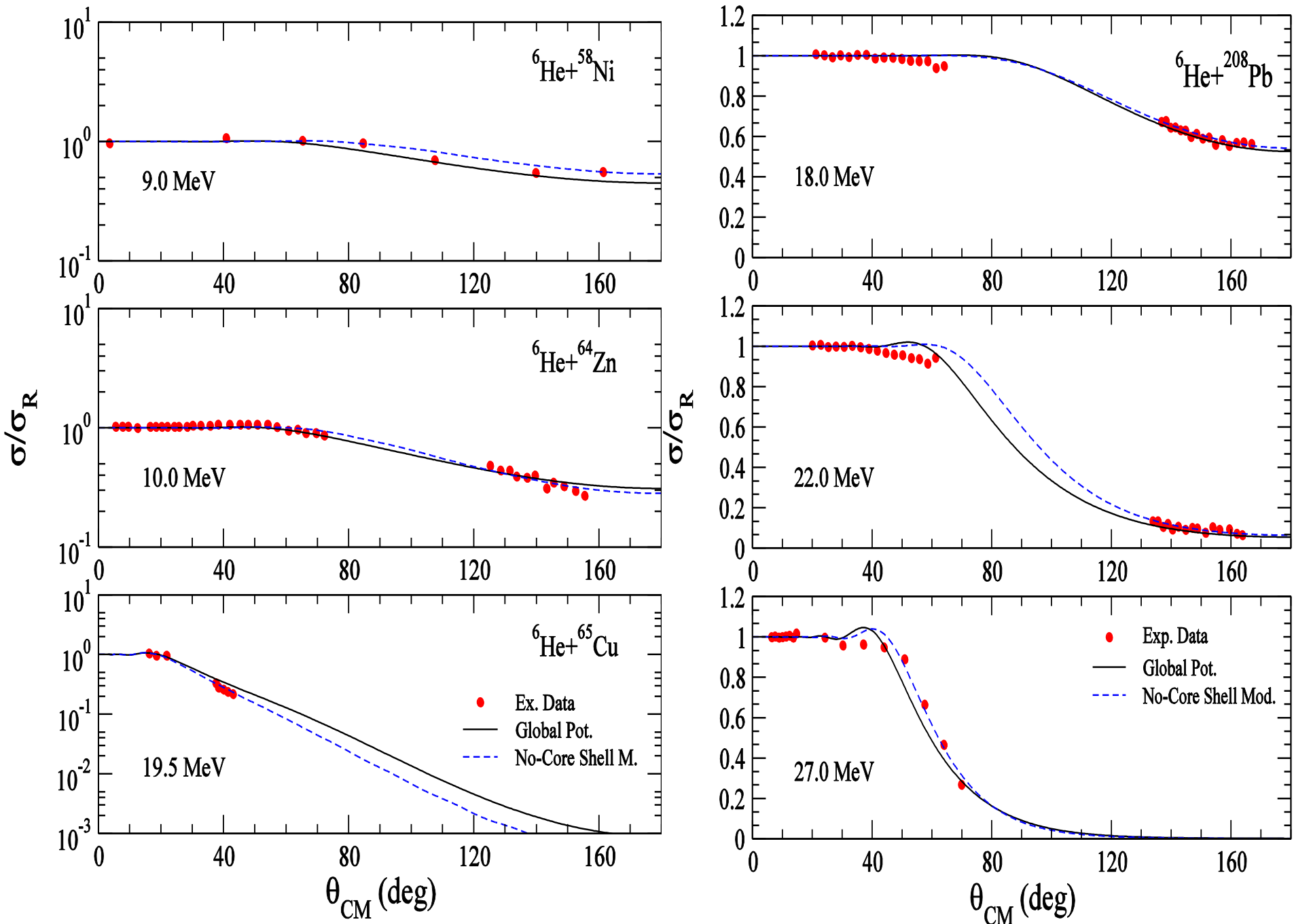
Imaginary & Real Potentials



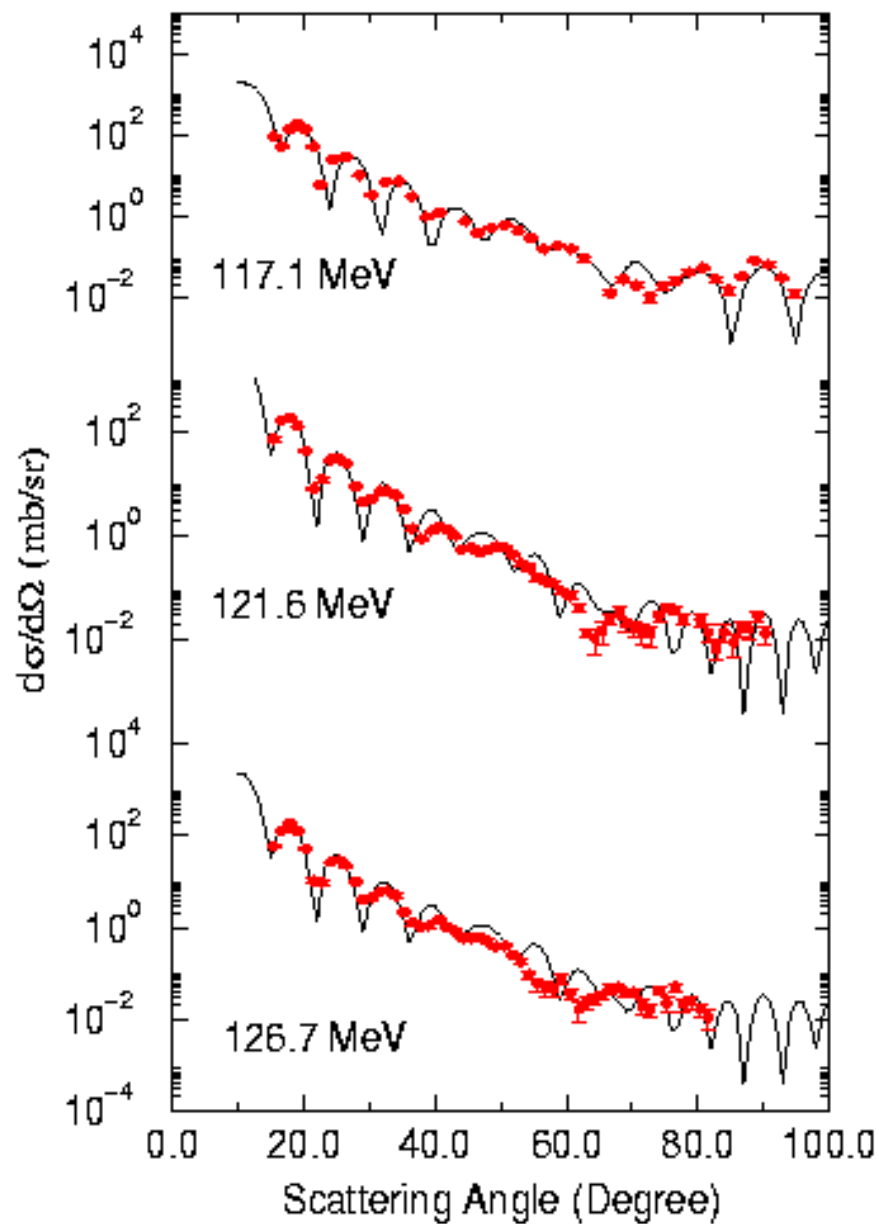
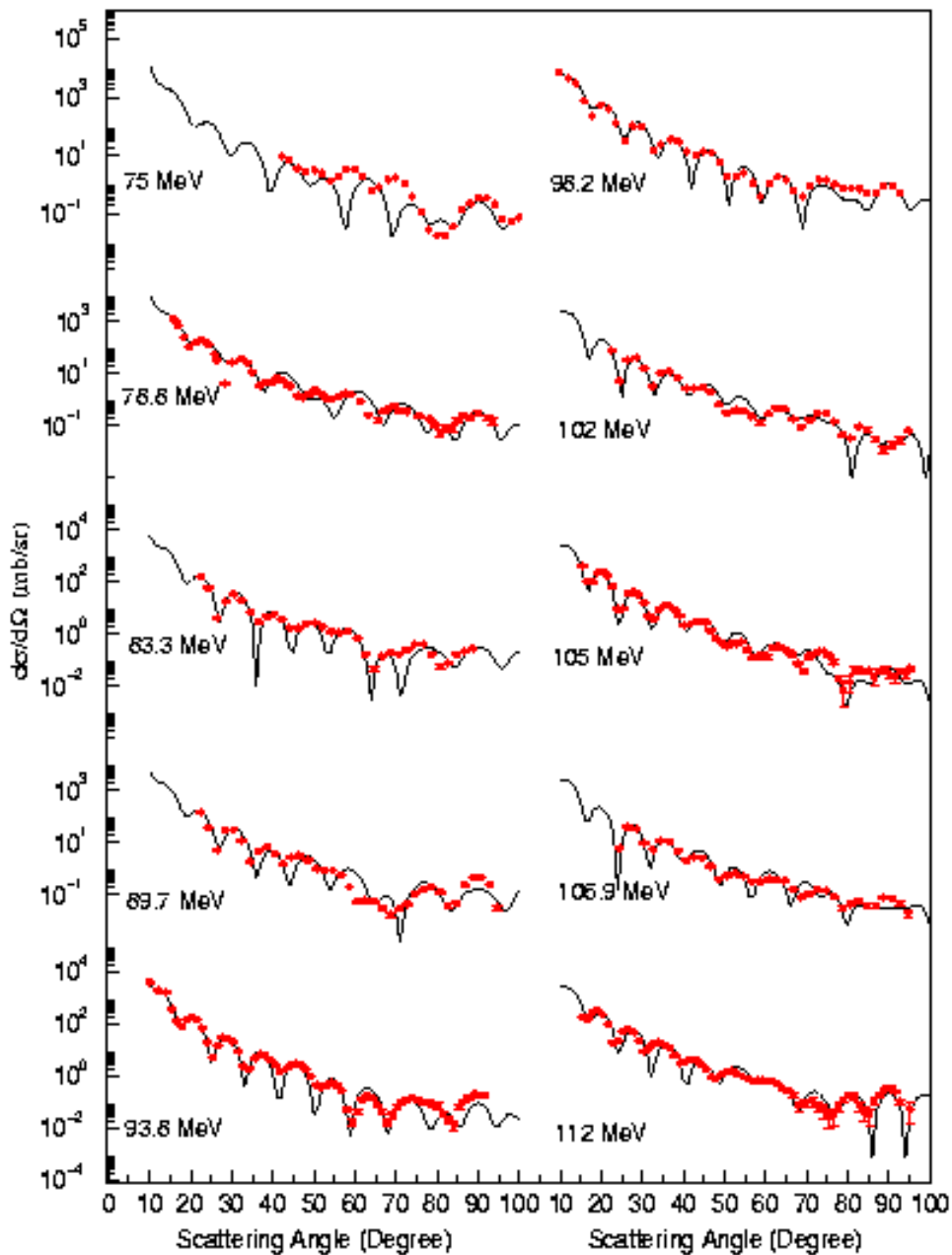
Long Range Absorption

Results

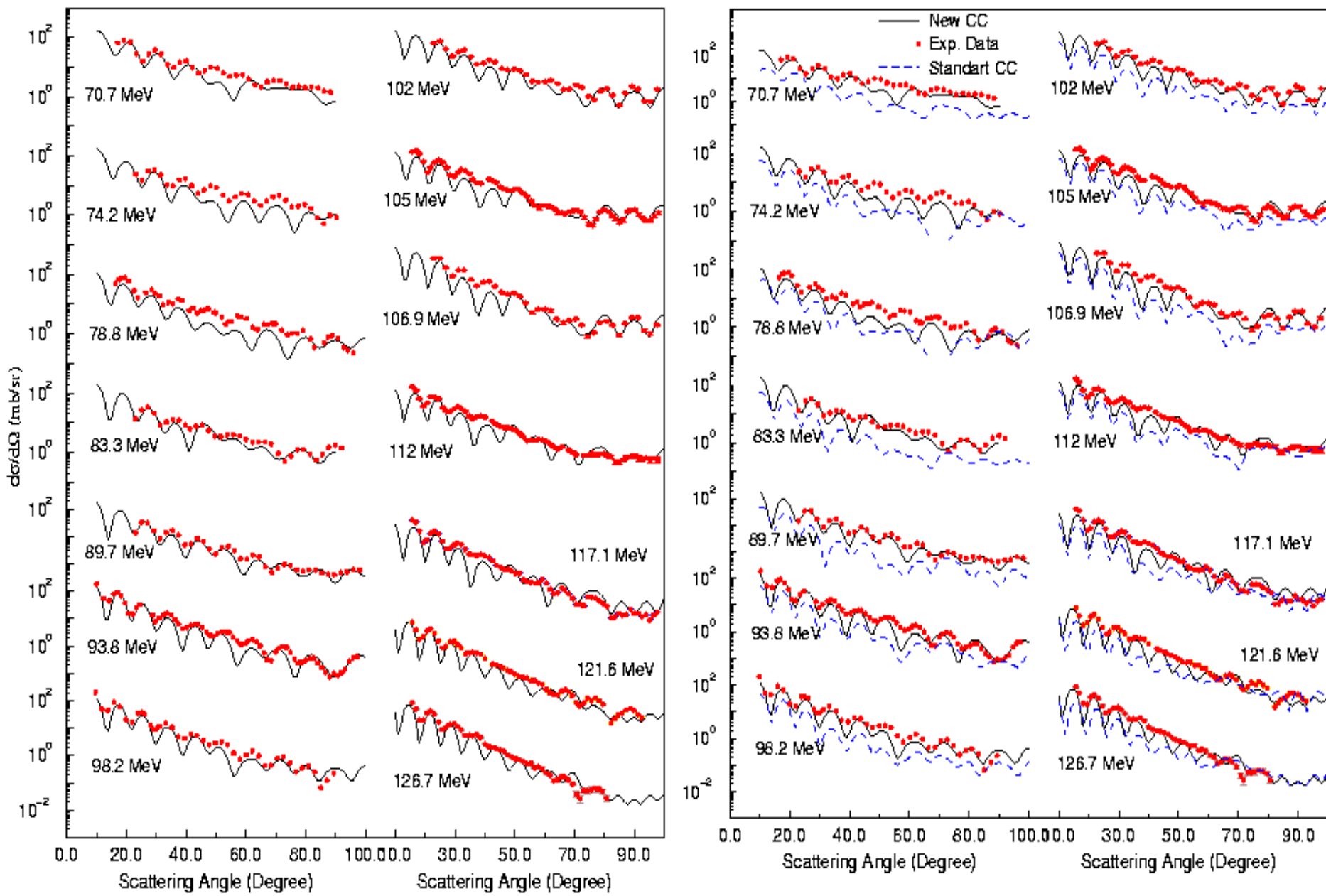




$^{12}\text{C}-^{12}\text{C}$



$^{12}\text{C}-^{12}\text{C}$, 32-127.5 MeV Single- 2^+



Thanks